Exotic Quantum Order in Low-Dimensional Systems

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Abstract

Strongly correlated quantum systems in low dimensions often exhibit novel quantum ordering. This ordering is sometimes hidden and can be revealed only by examining new 'dual' types of correlations. Such ordering leads to novel collective modes and fractional quantum numbers. Examples will be presented from quantum spin chains and the quantum Hall effect.

Key words: quantum Hall effect, fractional quantum Hall effect, magnetically ordered materials, spin dynamics

1 Introduction

The last two decades have witnessed remarkable experimental discoveries of novel quantum phenomena in strongly correlated systems in low dimensions. Along with this has come a vast increase in our understanding of how to describe the underlying order in these systems which often remains hidden to ordinary probes and has required the invention of new types of order parameters. As examples of this I will discuss here the Haldane gap in quantum spin chains, the Laughlin gap in the fractional quantum Hall effect and novel types of ordering in quantum Hall ferromagnets.

The underlying theme will be that a state which appears disordered in one representation can appear ordered in a dual representation. The word duality will be used in the sense of ordinary 'wave–particle' duality in quantum mechanics, and in the technical field-theoretic sense referring to a transformation which interchanges particles and vortices, or interchanges charge and flux.

2 Quantum Spin Chains

Let us begin with the idea of 'wave–particle' duality for quantum spins. Quantum spins are strange objects which behave in some ways like ordinary classical angular momentum vectors which can point in arbitrary directions. Yet if we measure their projection along any given direction, we find that it can take on only a discrete set of $2S+1$ values. The discreteness suggests that a collection of quantum spins might exhibit discrete excitations and a gap. The continuous orientation picture suggests a continuous spectrum of gapless spin waves. To be specific, let us consider the case of spin 1/2. The two possible discrete states can be represented by the spinors $|\uparrow\rangle$, and $|\downarrow\rangle$. However, quantum mechanics allows for the possibility of a coherent linear superposition of these two states

$$|\psi\rangle = \cos(\theta/2)e^{i\varphi/2}|\uparrow\rangle + \sin(\theta/2)e^{-i\varphi/2}|\downarrow\rangle. \tag{1}$$

This continuous family of states is parameterized by the two Euler angles determining the orientation of the expectation value of the spin vector.

Spin 1/2 is the least classical possible spin value, and yet the continuous spin wave picture works extremely well in dimensions greater than one. In the antiferromagnet the staggered magnetization

$$\vec{M}_s \equiv \sum_j (-1)^j \langle \vec{S}_j \rangle \tag{2}$$

is non-zero in the ground state (for $d > 1$). There exists a continuous family of spin wave excited states in which the Euler angles precess at a frequency linearly proportional to the wave vector (at long wavelengths). The same holds true for all possible other spin values $S = 1, 3/2, 2, \ldots$ They are all effectively equivalent and the discreteness of the individual spin orientations is irrelevant to the low energy physics.

The situation is very different in one dimensional spin chains where quantum fluctuations destroy the long range staggered magnetic order even at zero temperature. Half-integer spins $S = 1/2, 3/2, 5/2, \ldots$ still have gapless excitations, but they are *not* spin waves. It turns out that the spin wave, which would normally represent a $\Delta S = 1$ excitation above the singlet ground state, splits up ('fractionalizes') into two independent fermion-like objects known as 'spinons'. These objects are actually domain walls in the staggered order as shown in Fig.(1). This process is readily detectable experimentally because an ordinary spin wave is undamped at long wavelengths and so has a single sharply defined energy associated with each wavelength. A pair of spinons on the other hand has much more phase space available to it and the spectral density is a convolution of the spectral densities of the individual spinons.

Neutron scattering data demonstrating the existence of spinons is presented in Ref. [1].

For integer spin values $S = 1, 2, 3, \ldots$, the discreteness becomes relevant and the spin chain exhibits the so-called Haldane excitation gap. This can be understood by means of the following construction which relates the $S = 1$ chain to a dimerized $S = 1/2$ chain. Fig.(2) shows a spin chain in which the exchange coupling alternates between two different values J_1 and J_2 with $J_1 > J_2$. In order to capture the essence of the ground state let us consider the special case $J_2 = 0$. Here we know the exact ground state consists of independent singlets on the solid bonds illustrated in Fig.(2). This is a 'valence bond solid'. It has a non-degenerate ground state and a highly degenerate first excited state in which (any) one of the singlets is converted into a triplet. It costs a finite energy to break the singlet bond and so there is an excitation gap. If we now turn the second coupling J_2 back on, the valence bonds begin to fluctuate but the essential character of the state remains unchanged. The singlets are more likely to be found on the J_1 links, and the excitation gap is stable against the J_2 perturbation. The gap remains open for any value of $J_2 < J_1$ as illustrated in Fig.(3).

Fig.(3) also shows that the dimerization gap remains open even as the value of J_2 passes through zero and becomes negative. In the limit $J_2 \longrightarrow -\infty$, the pairs of spins on the J_2 links are forced to become parallel and their resulting triplet becomes an effective spin one degree of freedom, as shown in Fig.(4). The dimerization gap has now become the Haldane gap for an integer spin chain. The resulting 'valence bond solid' is an apparently featureless gapped spin 'liquid' state. Because of the excitation gap Δ, the local spin correlation function decays exponentially in (imaginary) time

$$\langle S_j^z(\tau) S_j^z(0) \rangle \sim e^{-\tau/\Delta}. \tag{3}$$

Similarly, the equal-time spin correlation function decays exponentially in space

$$\langle S_j^z S_{j+r}^z \rangle \sim e^{-r/\xi}. \tag{4}$$

This apparent featurelessness obscures an underlying hidden order which is characteristic of the valence bond solid state. It turns out that if one examines a typical configuration of the spins in the ground state it looks like this:

$$+00-+0000-+-0+00-+-000+0- \tag{5}$$

Notice that if we ignore the spins with $S^z = 0$, the remaining spins have perfect [2] antiferromagnet order,

$$+ - + - + - + - + - + - + - + - \tag{6}$$

Because of the random number of $S^z = 0$ sites inserted between the $S^z = \pm 1$ sites, the long-range antiferromagnetic order is invisible to the ordinary spin-spin correlation function which, as noted above, decays exponentially. This 'hidden' order is however manifest in the non-local den Nijs 'string' correlation function [3–5] defined by

$$\mathcal{O}_s^z \equiv \langle S_j^z \exp\left\{ i\pi \sum_{k=j+1}^{j+r-1} S_k^z \right\} S_{j+r}^z \rangle. \tag{7}$$

This object exhibits long-range order in the Haldane gapped phase

$$\lim_{r \longrightarrow \infty} \mathcal{O}_s^z \neq 0. \tag{8}$$

Breaking this topological order to create an excitation costs a finite energy gap.

Because of the excitation gap, the spin degrees of freedom disappear at low temperature and the magnetic susceptibility becomes exponentially small. One of the remarkable and paradoxical features of the Haldane phase is that introduction of *nonmagnetic* impurities actually liberates spin degrees of freedom. These are not, as one might have naively expected, $\Delta S = 1$ excitations but rather pairs of *fractional spin* $\Delta S = 1/2$ excitations. The process is illustrated in Fig.(5). Recall that the spin 1 on each site is viewed as a pair of spin 1/2 objects bound ferromagnetically into a triplet. As noted earlier, the valence bond solid ground state (in the absence of disorder) has each of the $S = 1/2$ objects pairing into a singlet bond with one of the spin 1/2 objects on the neighboring sites. (This is done in a symmetric way so that each site has total spin precisely equal to unity.) If one of the neighbors is a non-magnetic impurity, one of the singlet bonds is broken and *one-half* of the spin on the site is liberated. These weakly interacting fractional spins remain free to low temperatures and produce a Curie-like power law tail in the magnetic susceptibility. This is an example of a quantum McCoy-Griffiths singularity in which the den Nijs order (may) still be present but the gap is not [6,7].

3 Fractional Quantum Hall Effect

The fractional quantum Hall effect [8–11] (FQHE) is easily one of the richest and most remarkable phenomena discovered in many years. It turns out that there are deep analogies to the $S = 1$ spin chains we have just examined. Instead of an apparently featureless gapped *spin* liquid, we have an apparently featureless gapped *charge* liquid. As we shall see there is also a close analog of the hidden non-local string order. Breaking this order with a topological defect liberates a fractional *charge* (and in some cases spin as well).

For simplicity, let us focus on the $\nu = 1/3$ Hall plateau characterized by conductivity $\sigma_{xx} = 0, \sigma_{xy} = \frac{1}{3}\frac{e^2}{h}$. The Hamiltonian for the two-dimensional electron gas is

$$H = \sum_j \frac{1}{2m}\left(\vec{p}_j + \frac{e}{c}\vec{A}_j\right)^2 + \frac{1}{2}\sum_{i<j} v(\vec{r}_i - \vec{r}_j), \tag{9}$$

where $\vec{A}_j \equiv \vec{A}(\vec{r}_j)$ is the vector potential for the strong external magnetic field. To discover the hidden order in the Laughlin ground state we must make a singular gauge transformation which attaches three flux quanta to each electron. The Aharonov-Bohm phases associated with these flux tubes introduces an extra minus sign upon particle exchange and converts these composite objects into bosons [12–15]. The physics is completely identical in this representation, but the Hamiltonian changes to

$$H_{\mathrm{CB}} = \sum_j \frac{1}{2m}\left(\vec{p}_j + \frac{e}{c}(\vec{A}_j + \vec{a}_j)\right)^2 + \frac{1}{2}\sum_{i<j} v(\vec{r}_i - \vec{r}_j), \tag{10}$$

where

$$\vec{\nabla} \times \vec{a}_j = -\sum_{\ell \neq j} 3\Phi_0 \delta^2(\vec{r}_j - \vec{r}_l) \tag{11}$$

is the attached flux and Φ_0 is the flux quantum. The sign of the attached flux has been chosen so that on the average, the attached flux cancels the external field

$$\langle \vec{\nabla} \times (\vec{a}_j + \vec{A}_j) \rangle = 0. \tag{12}$$

Thus in the mean field approximation we have composite bosons moving in zero magnetic field. It is the condensation of these bosons that is the essential ordering in the quantum Hall effect. Because these objects behave as if

they carry 2D Coulomb charge, they exhibit algebraic (rather than true) off-diagonal long-range order

$$\langle \psi_{\rm CB}(\vec{r}) \psi^\dagger_{\rm CB}(\vec{r}\,') \rangle \sim |\vec{r} - \vec{r}\,'|^{-3/2}. \tag{13}$$

If we undo the singular gauge transformation to express this correlation function in terms of the original fermions, we have

$$\langle \psi_{\rm F}(\vec{r}) e^{i \int_{\vec{r}}^{\vec{r}\,'} d\vec{R} \cdot \vec{a}(\vec{R})} \psi^\dagger_{\rm F}(\vec{r}\,') \rangle. \tag{14}$$

Because, as can be seen from eq.(11), $\vec{a}$ depends on the position of *all* the particles, this non-local object is closely analogous to the 'string' correlator that describes the hidden order in integer spin chains.

Let us now examine the nature of topological defects (vortices) in the composite boson condensate. It turns out that because the objects obey an effective 2D electrodynamics (with logarithmic interactions among the charges), we expect flux quantization as in a superconductor

$$\int d^2r \, \vec{\nabla} \times (\vec{a} + \vec{A}) = \pm \Phi_0. \tag{15}$$

However because of the flux attachment transformation we have

$$\vec{\nabla} \times (\vec{a} + \vec{A}) = 3\Phi_0 \delta\rho \tag{16}$$

where $\delta\rho$ is the density deviation away from the mean. Hence, flux quantization implies that vortices carry fractional charge [12]

$$\int d^2r \, \delta\rho = \pm \frac{1}{3}, \tag{17}$$

and thus are the Laughlin quasiparticles. These are analogous to the fractional spin excitations appearing at the ends of $S = 1$ chains. (A more precise analogy can be made with the gapless excitations at the edges of quantum Hall liquids, which are in a sense a gas of Laughlin quasiparticles liberated at the edge [16,17].)

Another regime occurs at filling factor $\nu = 1/2$. This can the analyzed by attaching two rather than three flux quanta to the electrons, converting them not into composite bosons but rather composite *fermions* moving in zero average magnetic field. [18–21] While there are strong fluctuation corrections to this

mean field picture which are not yet fully understood, the basic phenomenology suggested by the picture has received striking experimental confirmation. [22,23]

4 Quantum Hall Ferromagnets

We turn now to the subject of ferromagnetism in quantum Hall systems, an area where significant advances have been recently made [24]. The large external magnetic field couples very strongly to the orbital motion and quenches the kinetic energy into discrete Landau levels. It turns out that the external field couples rather weakly to the spin degrees of freedom and so low energy spin fluctuations are not completely frozen out by the (small) Zeeman splitting.

We will consider here the case of a single filled Landau level. The ground state is fully spin polarized because this makes the spatial wave function antisymmetric which minimizes the Coulomb energy. We can describe the spin configuration in low-lying excited states buying introducing a unit vector field $\vec{m}(\vec{r})$ to describe the local spin orientation. The energy for slowly varying spin textures must then be of the form

$$U = \frac{1}{2}\rho_s \int d^2r\, \partial_\mu m^\nu \partial_\mu m^\nu - h \int d^2r\, m^z. \tag{18}$$

The spin stiffness $\rho_s \sim 5$K is a measure of how strongly the exchange energy prefers for the spins to be locally parallel and $h \sim 2$K represents the Zeeman coupling.

With this energy functional, the system has neutral bosonic spin waves with quadratic dispersion. These spin waves represent small oscillations of the spin orientation about the average direction. Somewhat surprisingly, there exist [25,26] topological defects in the underlying vector boson field $\vec{m}(\vec{r})$ which are actually fermions (for the case of filling factor $\nu = 1$). (There is an analogy here with the fermionic domain walls in the $S = 1/2$ spin chains.) These charged objects are called skyrmions by analogy with the topological defects which represent the nucleons in the Skyrme model of nuclear physics. Fig. (6) shows an example of such a spin texture. The spins are all pointing up at infinity and rotate smoothly downward towards the origin. At intermediate distances the spins lie in the XY plane and have a vortex-like arrangement. Such strange objects exist in ordinary two-dimensional ferromagnets (such as an iron film) but they are energetically expensive and freeze out at low temperatures. What is unique about the quantum Hall ferromagnet is that it is an itinerant magnet with a precisely quantized value of the Hall coefficient. It turns out that this causes the skyrmions to carry a precisely quantized

charge and fermion number. Thus by adding or subtracting charge from the filled Landau level, one can force skyrmions to exist even in the ground state of the system.

The size of a skyrmion is controlled by a competition between the Zeeman energy which wants to shrink the number of over turned spins, and the Coulomb Hartree energy which wants to spread the extra charge out over a large area. At ambient pressure in GaAs, a skyrmion contains about four flipped spins [27]. Under high pressure the effective g factor is driven towards zero and the skyrmion spin can be as large as 30 [28]. The large ratio of spin to charge of skyrmions has been directly measured using NMR [27], tilted field transport measurements [29] and optically [30].

In an ordinary ferromagnet the spin waves have a Zeeman gap which is much larger than the characteristic nuclear Zeeman splitting. Hence it is impossible for the nuclei to emit or absorb electronic spin waves. This situation is altered dramatically in the presence of skyrmions. The ground state of a quantum Hall ferromagnet slightly away from filling factor $\nu = 1$ is a lattice of skyrmions [31,32]. This means that the system has non-colinear order. Associated with this is a new $U(1)$ degree of freedom in which the spins rotate about the field direction. In a colinear magnet this does not produce an excitation because all the spins are aligned with the field. Because skyrmions contain spins lying in the XY plane, such rotations do represent new states. Since the rotation is about the field direction, this new collective mode does not have a Zeeman gap, but rather is a gapless Goldstone mode much like that in a superfluid. The analog of the phase of the superfluid order parameter is the local azimuthal orientation angle of the skyrmions. The analog of the conjugate boson number is the local number of flipped spins.

Because this new collective mode is gapless it couples very strongly to the nuclei and increases the relaxation rate $1/T_1$ by a factor of 10^3 over the zero magnetic field value. This dramatic enhancement brings the nuclei into thermal equilibrium which increases the specific heat [33] by factors exceeding 10^5.

5 Double-Layer Quantum Hall Systems

Recent technological progress has allowed the construction of double-layer quantum Hall systems [29] and related wide single well systems [34] in which two high-mobility electron gases are separated by only about $200\mathring{A}$, a distance comparable to the spacing between electrons. Inter-layer correlations can therefore be comparable to intra-layer correlations. We have already seen that Coulomb exchange effects can produce ferromagnetism. In double layer

systems there is a remarkable analog of this effect involving inter-layer correlations. Even if there is no tunneling allowed between the layers, quantum mechanics permits the possibility that we are uncertain which layer an electron is in. To visualize the significance of this, it is useful to define an 'isospin' which is up when the electron is in the left layer and down when the electron is in the right layer. A coherent superposition of these two possibilities is allowed just as is given in eq.(1). This of course represents an isospin state oriented in a direction given by the angles θ and φ. Just as for ordinary spin, Coulomb exchange effects prefer for the isospin vectors to be locally parallel. Because the inter-layer interaction is slightly weaker than the intra-layer interaction, the energy is not fully isospin rotation invariant but rather has the form

$$U = \frac{1}{2}\bar{\rho}_s \int d^2r\, \partial_\mu m^\nu \partial_\mu m^\nu + \int d^2r\, \lambda (m^z)^2. \tag{19}$$

The second term is an easy-plane anisotropy associated with the capacitive charging energy. If $m^z = \pm 1$, all of the electrons are in only one of the layers and there is a large Coulomb cost. This easy-plane anisotropy makes the system into an XY model which has a Kosterlitz-Thouless phase transition into an ordered state at low temperatures. The long range order of this state has been observed experimentally through the extreme sensitivity of the transport to small tilts of the applied magnetic field [29,34,9]. The Kosterlitz-Thouless transition (which has not yet been directly observed) is controlled by the unbinding of vortex-antivortex pairs, just as in a superconducting or superfluid films. As usual, an isolated vortex has a logarithmically divergent energy. In a superconducting film this is due to the kinetic energy stored in the supercurrent circulating around the vortex. In the present case it is the loss of Coulomb exchange energy associated with the isospin currents circulating around the vortex.

Recently a novel new type of broken symmetry involving a combination of isospin and real spin has been proposed [35] to explain certain features of the Raman spectrum in these systems [36].

6 Summary

In summary, we have seen that low-dimensional quantum systems can exhibit novel types of order. In some cases this order is hidden so that the system appears to be a featureless quantum liquid. However there often exists a dual picture in which the system order appears naturally. This dual picture can be achieved by interchanging the roles of charge and flux, particles and vortices, or waves and particles. We have also seen that the exotic ordering in low-dimensional quantum systems often produces new collective excitation modes

and fractional quantum numbers.

Acknowledgements

This work was supported by NSF DMR 97-14055. It is a pleasure to dedicate this paper to my long time friend and mentor, Eli Burstein, on the occasion of his 80th birthday.

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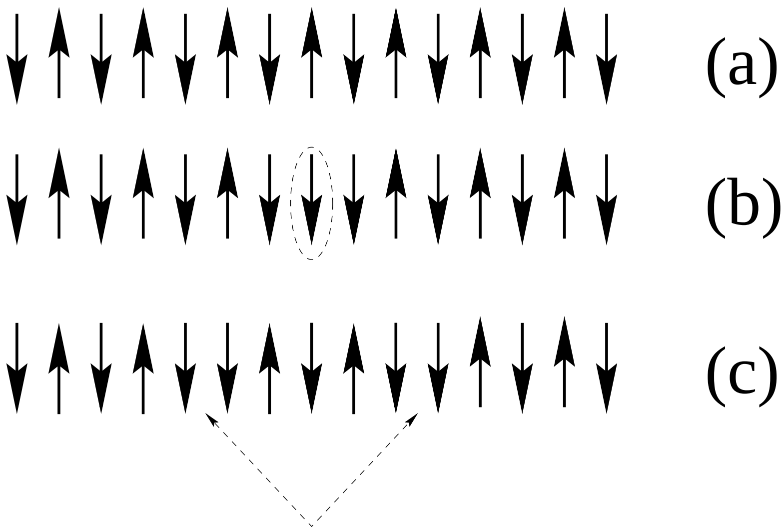

Fig. 1. (a) staggered ordered state in 1D. (b) excitation caused by flipping a single spin in the center of the chain. (c) fractionalization of the single flipped spin into two $S = 1/2$ domain walls (denoted by dashed arrows). The system moves from state (b) to state (c) by mutual spin flip of the pairs of spins on each side of the central spin.

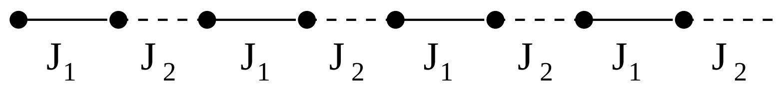

Fig. 2. Dimerized $S = 1/2$ chain with alternating couplings $J_1 > J_2$. The solid lines indicate singlet bonds.

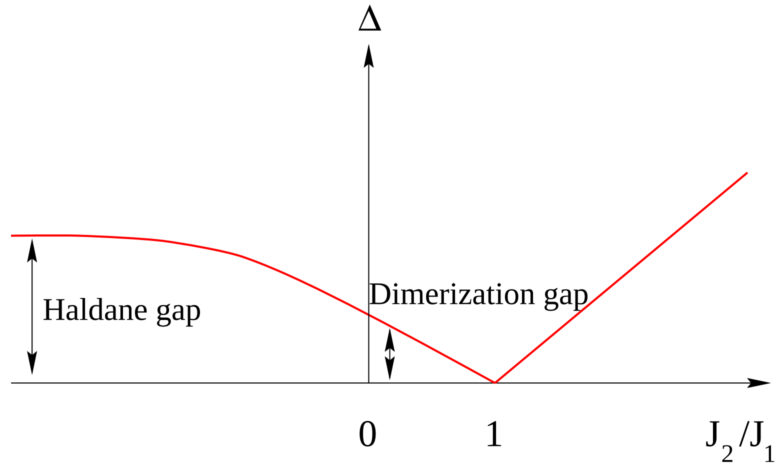

Fig. 3. Excitation gap Δ for a dimerized $S = 1/2$ chain as a function of the bond strength ratio J_2/J_1. The Haldane $S = 1$ chain corresponds to $J_2 \longrightarrow -\infty$.

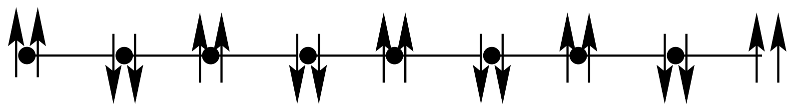

Fig. 4. Valence bond solid state in which the $S = 1$ spins on each site are represented as a symmetrical combination of two $S = 1/2$ objects. Each of these objects is paired into a singlet with one of its neighbors.

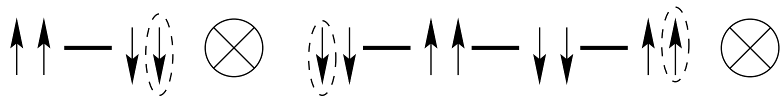

Fig. 5. Valence bond solid interrupted by non-magnetic impurities. The missing valence bonds liberate a *fractional* spin ($S = 1/2$) degree of freedom (indicated by the dashed ellipses) at the ends of each broken chain segment.

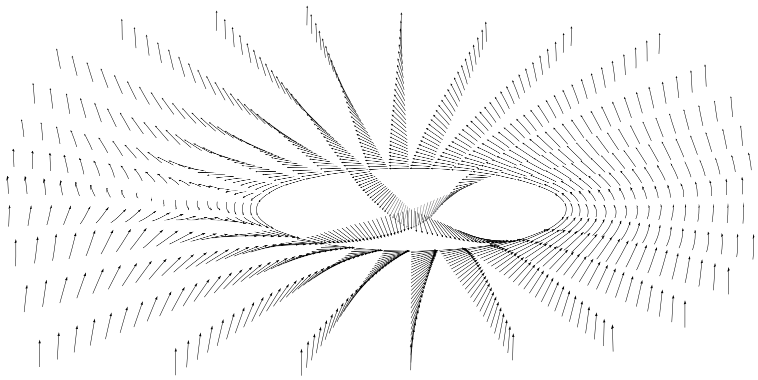

Fig. 6. A skyrmion spin texture in a quantum Hall ferromagnet. The spins are down at the origin but up at infinity. At intermediate distances the spins lie in the XY plane and have a vortex like configuration. Even though this object is an excitation of the bosonic spin field, the quantized Hall coefficient causes it to actually carry fermion number (and charge) equal to the quantum number $\nu \equiv \frac{h}{e^2}\sigma_{xy}$.